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Almost complex manifolds and Hirzebruch invariant for isolated singularities in complex spaces

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§1. Introduction

In [8], Hirzebruch has defined an invariant φ for normal two-dimensional singularities and calculated it explicitly for some cases.

In this paper, we generalize his procedure for higher dimensional isolated singularities. The definition can be given exactly the same way following Hirzebruch's idea. However we have to introduce a new technique to show the well-definedness. For Hirzebruch has used the existence of minimal resolution, but there may not be any minimal resolution in higher dimensional case.

Roughly speaking, the technique is a method to modify a given U-manifold (a smooth manifold with a complex vector bundle structure on the stable tangent bundle) to obtain an almost complex manifold without changing some particular invariants.

This technique will be expressed in §2 and by using it we will show

Theorem 2-4. Any element x in the complex bordism group Ω_{2n} $(n \neq 1)$ can be expressed by a connected almost complex manifold.

In §3, we will define an invariant φ for some kind of isolated singularities in complex spaces, which we may call the Hirzebruch invariant. Using the technique developed in §2, we will show that the definition is well-defined.

In §4, we will calculate the invariant φ for quotient singularities. (Theorem 4-1)

Finally in §5, we will consider some special cases, e.g. the Brieskorn singularities and the singularities whose "normal neighborhood manifold" is diffeomorphic to a homotopy sphere.

§2. Almost complex manifolds

Let M^{2n} be a smooth 2n-manifold and let $\tau(M)$ be the tangent bundle of M. Let $BU(n+m)$ (m : large) be the classifying space for complex (n+m)-dimensional vector bundles and let $\xi^{(n+m)}$ be the universal bundle over $BU(n+m)$. A U-structure on M is a real bundle map

$$b : \tau(M) \oplus \varepsilon^{2m} \longrightarrow \xi^{(n+m)}$$

where ε^{2m} is a trivial real vector bundle of dimension 2m. Two U-structures

$$b_i : \tau(M) \oplus \varepsilon^{2m} \longrightarrow \xi^{(n+m)} \quad (i = 0, 1)$$

are said to be (stably) equivalent if there is a real bundle map

$$B : (\tau(M) \oplus \varepsilon^{2m}) \times I \longrightarrow \xi^{(n+m)}$$

such that $B \mid M \times \{i\} = b_i$ $(i = 0, 1)$, where $(\tau(M) \oplus \varepsilon^{2m}) \times I$ is the induced bundle over $M \times I$.

Then as is well-known, there is a one to one correspondence between the set of equivalence classes of U-structures on M and the homotopy classes of liftings of the map $f_M : M \longrightarrow BSO$ to BU, where f_M is the classifying map for the stable tangent bundle of M (cf. [3] §4).

Similarly, an almost complex structure on M consists of a real bundle map

$$b : \tau(M) \longrightarrow \xi^{(n)}$$

where $\xi^{(n)}$ is the universal complex bundle over $BU(n)$. Two almost complex structures

$$b_i : \tau(M) \longrightarrow \xi^{(n)} \quad (i = 0, 1)$$

are said to be equivalent if there is a bundle map

$$B : \tau(M) \times I \longrightarrow \xi^{(n)}$$

such that $B \mid M \times \{i\} = b_i \quad (i = 0, 1)$.

Now we consider the following problem:

Let M^{2n} be a closed U-manifold. Then when does M admit an almost complex structure?

For this, we introduce an invariant $\lambda(M)$ by

$$\lambda(M) = \frac{1}{2}(\chi(M) - c_n[M])$$

where $\chi(M)$ is the Euler number of M and $c_n[M]$ is the n-th Chern number of M. Since $c_n[M] \bmod 2 = w_{2n}[M] = \chi(M) \bmod 2$, $\lambda(M)$ is actually an integer.

We will prove

Proposition 2-1. (i) If $n \equiv 0 \pmod 2$, then M admits an almost complex structure, which is stably equivalent to the original U-structure on $M - \mathring{D}^{2n}$ if and only if

$$\lambda(M) = 0 ,$$

where D^{2n} is a disk in M.

(ii) If $n \equiv 1 \pmod 4$, then M admits an almost complex structure, which is stably equivalent to the original U-structure on $M - \mathring{D}^{2n}$ if and only if

$$\lambda(M) \equiv 0 \pmod{(n-1)!}.$$

(iii) If $n \equiv 3 \pmod 4$, then M admits an almost complex structure, which is stably equivalent to the original U-structure on $M - \mathring{D}^{2n}$ if and only if

$$\lambda(M) \equiv 0 \pmod{\frac{(n-1)!}{2}}.$$

(iv) In any case, if $\lambda(M) = 0$, then M admits an almost complex structure which is stably equivalent to the original U-structure.

Remark. Massey [10] has considered a more general problem. In fact our Prop. 1-1, (i) for $n \equiv 2 \pmod 4$ can be found in his Theorem II.

Before proving Proposition 2-1, we prepare the following well known lemma.

Lemma 2-2. (i) Homotopy groups $\pi_{2n-1}(SO(2n)/U(n))$ are given as follows.

$$\pi_{2n-1}(SO(2n)/U(n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0 \pmod 4 \\ \mathbb{Z}/(n-1)! & n \equiv 1 \pmod 4 \\ \mathbb{Z} & n \equiv 2 \pmod 4 \\ \mathbb{Z}\Big/\frac{(n-1)!}{2} & n \equiv 3 \pmod 4 \end{cases}$$

where the summand $\mathbb{Z}$ for the case $n \equiv 0 \pmod 4$ is

$$\mathrm{Ker}(i_* : \pi_{2n-1}(SO(2n)/U(n)) \longrightarrow \pi_{2n-1}(SO/U)).$$

(ii) There is a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, i_* \longrightarrow \pi_{2n}(BSO(2n)) \xrightarrow{i_*} \pi_{2n}(BSO) \longrightarrow 0$$

and $\mathrm{Ker}\, i_*$ is isomorphic to $\mathbb{Z}$ generated by the tangent bundle of S^{2n} and detected by $\frac{1}{2}\chi$, where χ is the Euler class.

(iii) Let $\partial : \pi_{2n}(BSO(2n)) \longrightarrow \pi_{2n-1}(SO(2n)/U(n))$ be the boundary homomorphism associated with the following fibration

$$SO(2n)/U(n) \longrightarrow BU(n) \longrightarrow BSO(2n).$$

Then ∂ is an epimorphism and ∂ carries the summand

$\operatorname{Ker} i_* \subset \pi_{2n}(BSO(2n))$ onto the summands $\mathbb{Z}$, $\mathbb{Z}/(n-1)!$ or $\mathbb{Z}\Big/\frac{(n-1)!}{2}$ according as $n \equiv 0$ or $2 \pmod 4$, $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ respectively.

Proof of Proposition 2-1. Let M^{2n} be a connected U-manifold. From homotopy theoretical point of view this means the following. Let $f_M : M \longrightarrow BSO(2n)$ be the classifying map for the tangent bundle of M and let $i^*(BU)$ be the bundle over $BSO(2n)$ induced by the natural map $i : BSO(2n) \longrightarrow BSO$;

$$\begin{tikzcd} & i^*(BU) \arrow[r] \arrow[d] & BU \arrow[d] \\ M \arrow[ur] \arrow[r, "f_M"] & BSO(2n) \arrow[r, "i"] & BSO . \end{tikzcd}$$

Then there is given a lifting $\tilde{f}_M : M \longrightarrow i^*(BU)$ of f_M. Now since $(SO/U,\ SO(2n)/U(n))$ is $(2n-1)$-connected, the lifting $\tilde{f}_M$ is homotopic (through fibre) to a map $\tilde{f}'_M$ such that $\tilde{f}'_M \mid M - \mathring{D}^{2n}$ factors through $BU(n) \subset i^*(BU)$. This means the following.

There is real vector bundle isomorphisms

$$b : \tau(M) \oplus \varepsilon^{2m} \longrightarrow \xi^{(n+m)}$$

$$b' : \tau(M) \mid M - \mathring{D} \longrightarrow \xi^{(n)}$$

making the following diagram commutative,

$$\begin{tikzcd} \tau(M) \mid M-\mathring{D} \oplus \varepsilon^{2m} \arrow[r, "b'\oplus \varepsilon"] \arrow[dr, "b \mid M-\mathring{D}"'] & \xi^{(n)} \oplus \varepsilon^{(m)} \arrow[d, "b_{n,n+m}"] \\ & \xi^{(n+m)} \end{tikzcd}$$

where ε and $b_{n,n+m}$ are natural bundle maps. Moreover b is equivalent to the original U-structure on M.

Thus there is a complex vector bundles $\mathcal{J}^{(n+m)}$ over M, $\bar{\mathcal{J}}^{(n)}$ over $M - \mathring{D}$, real bundle isomorphisms

$$d_1 : \mathcal{J}^{(n+m)} \longrightarrow \tau(M) \oplus \varepsilon^{2m}$$
$$d_2 : \bar{\mathcal{J}}^{(n)} \longrightarrow \tau(M) \mid M - \mathring{D}$$

and a complex vector bundle isomorphism

$$d : \bar{\mathcal{J}}^{(n)} \oplus \varepsilon^{(m)} \longrightarrow \mathcal{J}^{(n+m)} \mid M - \mathring{D}$$

making the following diagram commutative,

$$\begin{tikzcd}
\bar{\mathcal{J}}^{(n)} \oplus \varepsilon^{(m)} \arrow[rr, "d"] \arrow[dr, "d_2 \oplus \varepsilon"'] & & \mathcal{J}^{(n+m)} \mid M - \mathring{D} \arrow[dl, "d_1 \mid M - \mathring{D}"] \\
& \tau(M) \mid M - \mathring{D} \oplus \varepsilon^{2m} &
\end{tikzcd}$$

where $\varepsilon : \varepsilon^{(m)} \longrightarrow \varepsilon^{2m}$ is a canonical real bundle isomorphism.

Now choose a framing

$$\psi : \tau(M) \mid D^{2n} \longrightarrow \mathbb{R}^{2n} .$$

This gives a framing

$$\psi \mid S^{2n-1} : \tau(M) \mid S^{2n-1} \longrightarrow \mathbb{R}^{2n} .$$

where $S^{2n-1} = \partial D^{2n}$. On the other hand, there is given a complex vector space structure $\bar{\mathcal{J}}^{(n)} \mid S^{2n-1}$ on $\tau(M) \mid S^{2n-1}$. Thus we have obtained a "SO(2n)/U(n)-bundle" α over S^{2n-1};

$$\alpha \in \pi_{2n-1}(SO(2n)/U(n))$$

and it is clear that the almost complex structure $\bar{\mathcal{J}}^{(n)}$ on $M - \mathring{D}$ extends to whole of M if and only if $\alpha = 0$. Now since $\partial : \pi_{2n}(BSO(2n)) \longrightarrow \pi_{2n-1}(SO(2n)/U(n))$ is epimorphic, we can choose an element γ in $\pi_{2n}(BSO(2n))$ such that $\partial(\gamma) = \alpha$. We choose such an element γ as follows.

Since $\pi_{2n-1}(BU(n)) = 0$, the complex vector bundle $\bar{\mathcal{J}}^{(n)} \mid S^{2n}$ is trivial. Let

$$\eta : \mathcal{J}^{(n+m)} \mid D^{2n} \longrightarrow \mathbb{C}^{n+m}$$

be a framing. By restricting, we obtain a framing

$$\eta | S^{2n-1} : \mathcal{J}^{(n+m)} | S^{2n-1} \longrightarrow \mathbb{C}^{n+m}.$$

But since $\mathcal{J}^{(n+m)} | S^{2n-1} \cong \bar{\mathcal{J}}^{(n)} | S^{2n-1} \oplus \varepsilon^{(m)}$ and

$$\pi_{2n-1}(U(n)) \xrightarrow{\sim} \pi_{2n-1}(U(n+m)),$$

there is a framing

$$\bar{\eta} : \bar{\mathcal{J}}^{(n)} | S^{2n-1} \longrightarrow \mathbb{C}^{(n)}$$

such that $\bar{\eta} \oplus \varepsilon$ is homotopic to $\eta | S^{2n-1}$, where $\varepsilon : \varepsilon^{(m)} \longrightarrow \mathbb{C}^m$ is the natural framing. Now the framing $\bar{\eta}$ yields a complex vector bundle $\bar{\mathcal{J}}^{(n)}/\bar{\eta}$ over $M - \mathring{D}/S^{2n-1}$ and a real vector bundle $\tau(M) | D^{2n}/\bar{\eta}$ over $D^{2n}/S^{2n-1} = S^{2n}$. We set $\tau(M) | D^{2n}/\bar{\eta} = \gamma$. Then it is clear that $\partial(\gamma) = \alpha$. Moreover since the framing $\bar{\eta}$ stably extends to a framing over D^{2n}, it is clear that

$$\gamma \in \operatorname{Ker} i_*$$

where $i_* : \pi_{2n}(BSO(2n)) \longrightarrow \pi_{2n}(BSO)$.

Thus by Lemma 2-2 (ii), γ is detected by the Euler class χ.

Let

$$g : M \longrightarrow M - \mathring{D} / S^{2n-1} \vee D^{2n}/S^{2n-1}$$

be the natural map. Then clearly

$$\tau(M) = g^*(\bar{\mathcal{J}}^{(n)}/\bar{\eta} \vee \gamma).$$

Therefore

$$\chi(\tau(M)) = \chi(g^*(\bar{\mathcal{J}}^{(n)}/\bar{\eta} \vee \gamma)).$$

We evaluate this class on the fundamental class $[M]$.

$$\begin{aligned}\langle \chi(\tau(M)), [M] \rangle &= \langle \chi(g^*(\bar{\mathcal{J}}^{(n)}/\bar{\eta} \vee \gamma)), [M] \rangle \\ &= \langle \chi(\bar{\mathcal{J}}^{(n)}/\bar{\eta} \vee \gamma), g_*[M] \rangle \\ &= \langle \chi(\bar{\mathcal{J}}^{(n)}/\bar{\eta}), [M - \mathring{D} / S^{2n-1}] \rangle \\ &\quad + \langle \chi(\gamma), [S^{2n-1}] \rangle.\end{aligned}$$

But clearly

$$\langle \chi(\tau(M)),\ [M]\rangle = \chi(M).$$

Since the framing $\bar{\eta}\mid S^{2n-1}$ stably extends to the framing $\eta\mid D^{2n}$, we have

$$\begin{aligned}\langle \chi(\bar{\mathfrak{J}}^{(n)}/\bar{\eta}),\ [M-\mathring{D}/S^{2n-1}]\rangle &= \langle c_n(\bar{\mathfrak{J}}^{(n)}/\bar{\eta}),\ [M-\mathring{D}/S^{2n-1}]\rangle \\ &= \langle c_n(\mathfrak{J}^{(n+m)}),\ [M]\rangle \\ &= c_n[M].\end{aligned}$$

Thus we have obtained

$$\begin{aligned}\langle \chi(\gamma),\ [S^{2n}]\rangle &= \chi(M) - c_n[M] \\ &= 2\lambda(M).\end{aligned}$$

We first consider the case (i).

(i) $n \equiv 0 \pmod 2$. In this case, since $\partial : \mathrm{Ker}\, i_* \longrightarrow \pi_{2n-1}(SO(2n)/U(n))$ is injective, $\alpha = 0$ if and only if $\gamma = 0$. But since $\mathrm{Ker}\, i_*$ is detected by $\frac{1}{2}\chi$, $\gamma = 0$ if and only if

$$\tfrac{1}{2}\langle \chi(\gamma),\ [S^{2n}]\rangle = \lambda(M) = 0\ .$$

(ii) $n \equiv 1 \pmod 4$. By Lemma 2-2, $\alpha = 0$ if and only if

$$\lambda(M) = \tfrac{1}{2}\langle \chi(\gamma),\ [S^{2n}]\rangle \equiv 0 \pmod{(n-1)!}.$$

(iii) $n \equiv 3 \pmod 4$. Similarly we have,

$$\alpha = 0 \text{ if and only if } \lambda(M) \equiv 0 \left(\mathrm{mod}\ \frac{(n-1)!}{2}\right).$$

(iv) is clear from the above argument.

(Q.E.D.)

As a corollary to Proposition 2-1, we obtain the following well known theorem.

Theorem 2-3 (Borel, Serre [2]). Let Σ^{2n} be a homotopy sphere. Then Σ^{2n} admits an almost complex structure if and only if $n = 1$ or 3.

Proof. Since Σ^{2n} is a π-manifold, it has a trivial

U-structure. With this structure, our invariant λ is

$$\lambda(\Sigma^{2n}) = 1.$$

Now since $\Sigma^{2n} - \mathring{D}^{2n}$ is diffeomorphic to D^{2n}, any two almost complex structures on $\Sigma^{2n} - \mathring{D}^{2n}$ are equivalent. Thus by Proposition 2-1, Σ^{2n} admits an almost complex structure if and only if

$$n \equiv 1 \pmod 4 \quad \text{and} \quad 1 \equiv 0 \pmod{(n-1)!}$$

$$\text{or} \quad n \equiv 3 \pmod 4 \quad \text{and} \quad 1 \equiv 0 \pmod{\frac{(n-1)!}{2}}.$$

But this holds only for $n = 1$ or 3. (Q.E.D.)

Now we prove the following theorem, which is the main result of this section.

Theorem 2-4. Let Ω_{2n}^{U} be the complex bordism group. If $n \neq 1$, then any element x in Ω_{2n}^{U} can be represented by a connected almost complex manifold. If $n = 1$, then an element $x \in \Omega_2^U$ can be represented by a connected almost complex manifold if and only if the first Chern number $c_1(x) \leqq 2$.

Proof. The latter part is clear. Thus let x be any element in Ω_{2n}^{U} $(n \neq 1)$. We first remark that the connected sum operation works also well in weakly almost complex context. If $n \not\equiv 0 \pmod 4$, then this is clear, for $\pi_{2n-1}(SO/U) = 0$. But if $n \equiv 0 \pmod 4$, then $\pi_{2n-1}(SO/U) = \mathbb{Z}/2$. However it can be easily checked that the connected sum is well defined. Thus we may assume that x is represented by a connected U-manifold M. Now if $\lambda(M) = 0$, then by Proposition 2-1, (iv), M admits an almost complex structure, which is stably equivalent to the given U-structure. Therefore x satisfies the required condition. Thus we have only to modify M to kill our invariant $\lambda(M)$ without changing the Chern numbers.

Now let N be a connected π-manifold. We give a trivial U-structure on N. Then clearly

$$[M \# N] = [M].$$

We check how our invariant λ varies under the connected sum operation,

$$\lambda(M \# N) = \lambda(M) + \lambda(N) - 1.$$

Therefore we have only to prove the following lemma.

Lemma 2-5. Assume $n \neq 1$. Then for any integer $m \in \mathbf{Z}$, there is a connected π-manifold N^{2n} with $\lambda(N) = m$.

Proof. Let T^{2n} be the 2n-dimensional torus. Then clearly

$$\lambda(T^{2n}) = 0.$$

By induction, we obtain

$$\lambda(T^{2n} \underbrace{\# \cdots \# T^{2n}}_{(m+1)\text{-times}}) = -m.$$

Clearly

$$\lambda(S^{2n}) = 1$$

and

$$\lambda(S^2 \times S^{2n-2}) = 2.$$

(Here we use the fact $n \neq 1$.) By induction, we obtain

$$\lambda(S^2 \times S^{2n-2} \underbrace{\# \cdots \# S^2}_{(m-1)\text{ times}} \times S^{2n-2}) = m.$$

(Q.E.D.)

Thus there are enough π-manifolds to kill our invariant $\lambda(M)$. This proves Theorem 2-4. (Q.E.D.)

Finally we prove the following proposition for later use.

Proposition 2-6. Let M^{2n} be a connected U-manifold with non-empty boundary ∂M. Assume that there is given an almost complex structure on a neighborhood of ∂M, which is stably equivalent to the given U-structure. Then there is a closed U-manifold N such that $M \# N$ admits an almost complex structure which is equivalent to the given one on a neighborhood of ∂M.

Moreover we may assume the following. Let

$$p : M \# N \longrightarrow M$$

be the natural collapsing map. Then

$$p_* : \pi_1(M \# N) \xrightarrow{\sim} \pi_1(M).$$

Proof. The proof is similar to those of Proposition 2-1 and Theorem 2-4. Here we only check the last condition. Now if we use T^{2n} for N^{2n}, then the fundamental group π_1 may change. So we cannot use this. Instead we use the Riemann sphere $\mathbb{C}P^1$. Our invariant of $\mathbb{C}P^1$ with the natural complex structure is

$$\lambda(\mathbb{C}P^1) = 0.$$

Therefore, by induction

$$\lambda(\underbrace{\mathbb{C}P^1 \# \cdots \# \mathbb{C}P^1}_{(m+1)\text{-times}}) = -m.$$

(Compare with Lemma 2-5.) Thus there are enough closed U-manifolds whose fundamental group are zero. Therefore we can modify M to obtain an almost complex manifold $M \# N$ without changing the fundamental group. (Q.E.D.)

§ 3. Hirzebruch invariant for isolated singularities in complex spaces.

Let $L_n(p_1, p_2, \cdots, p_n) \in \mathbb{Q}[p_1, p_2, \cdots, p_n]$ be the Hirzebruch L-polynomial of degree n and let M^{4n} be a closed oriented smooth manifold. Then Hirzebruch's signature theorem [6] states that

$$\operatorname{sign} M = L[M]$$

where sign M is the signature of M and L[M] is the L-genus of M.

Now assume M^{4n} is a U-manifold of real dimension 4n. Then the Pontrjagin classes of M can be expressed in terms of the Chern classes of M. Thus let

$$\bar{L}_{2n}(c_1, c_2, \cdots, c_{2n}) \in \mathbb{Q}[c_1, c_2, \cdots, c_{2n}]$$

be the polynomial obtained from $L_n(p_1, p_2, \cdots, p_n)$ by substituting

$$p_k = (-1)^k \sum_{i+j=2k} (-1)^i c_i c_j \ .$$

Then

$$\operatorname{sign} M = \bar{L}[M] \ .$$

Now recall, for almost complex manifold M^{4n}, we have

$$c_{2n}[M] = \chi(M).$$

Let M^{4n} be an almost complex manifold of real dimension 4n with possibly non-empty boundary ∂M. Now assume that for any $0 < i < n$, the rational Chern class c_{2i} restricted to the boundary ∂M vanishes. Then we can define the "$\tilde{L}$-genus" of M, $\tilde{L}[M]$, as follows. First we recall

$$\bar{L}_{2n}(c_1, c_2, \cdots, c_{2n}) = \beta_n c_{2n} + \text{decomposable terms}$$

where

$$\beta_n = (-1)^n \frac{2^{2n+1}(2^{2n-1} - 1)B_n}{(2n)!} \ .$$

Let $xy \in H^{4n}(BU; \mathbb{Q})$ be a decomposable factor that appears in $\bar{L}_{2n}$. Thus $0 < \deg x = 2i < 4n$. By the hypothesis, we have $i^*x(M) = 0$, where

$$i^* : H^{2i}(M; \mathbb{Q}) \longrightarrow H^{2i}(\partial M; \mathbb{Q}).$$

Therefore $x(M) = j^{*}\tilde{x}(M)$ for some $\tilde{x}(M) \in H^{2i}(M, \partial M; \mathbb{Q})$. Now we define the Chern number $xy[M]$ by

$$xy[M] = \langle \tilde{x}(M)y(M), [M, \partial M] \rangle .$$

It is easy to verify that this does not depend on the choice of $\tilde{x}(M)$. Then we define the $\tilde{L}$-genus of M by

$$\tilde{L}[M] = \beta_n \chi(M) + \text{decomposable Chern numbers of } M.$$

Clearly if $\partial M = \emptyset$, then

$$\tilde{L}[M] = \operatorname{sign} M.$$

Now let $(V^{(2n)}, P)$ be a germ of complex analytic space at P of complex dimension $2n$ and assume that P is an isolated singular point of V. Fix a Lojasiewicz triangulation [9] K of V. Let M be the link of P in K. It is a closed PL-manifold of dimension $4n-1$. Moreover a punctured neighborhood of P is PL-homeomorphic to $M \times \mathbb{R}$. Since $V - \{P\}$ is a complex manifold, $M \times \mathbb{R}$ has a smooth structure $\mathcal{S}$. Therefore by Cairns-Hirsch theory, we can uniquely smooth M so that $M \times \mathbb{R}$ is diffeomorphic to $(M \times \mathbb{R})_{\mathcal{S}}$.

Now we choose V so that V is homeomorphic to the cone over $\partial V \underset{\text{diffeo}}{\cong} M$. We make the following definition.

Definition 3-1. An isolated singular point $P \in V$ is said to be rationally parallelizable if all the rational Chern classes of a punctured neighborhood of V vanish.

Now we define the Hirzebruch invariant φ.

Definition 3-2. Let $(V^{(2n)}, P)$ be a rationally parallelizable isolated singularity. Let $\pi : \tilde{V} \longrightarrow V$ be a resolution ([5]). By the definition of rational-parallelizability, we can define the $\tilde{L}$-genus of $\tilde{V}$, $\tilde{L}[\tilde{V}]$, and we put

$$\varphi(P) = \tilde{L}[\tilde{V}] - \text{sign } \tilde{V}.$$

Proposition 3-3. The above definition is well-defined, i.e. it does not depend on the choice of resolution $\tilde{V}$.

Proof. Consider the manifold $\partial V = \partial \tilde{V}$. It is a closed smooth (4n-1)-manifold with a U-structure induced from the complex manifold structure on $\tilde{V}$. Since $\Omega^U_{4n-1} = 0$, there is a U-manifold W^{4n} such that

$$\partial W = \partial V \text{ "from outside".}$$

Now there is an almost complex structure on a neighborhood of ∂W compatible with the U-structure of W. Therefore by Proposition 2-6, we can modify W in the interior to obtain an almost complex manifold $\tilde{W}$. Now let $M = \tilde{V} \underset{\partial \tilde{V}}{\cup} \tilde{W}$. It is an almost complex manifold. Therefore we have

(1) $$\text{sign } M = \tilde{L}[M].$$

By the Novikov lemma on the additivity of the signature,

(2) $$\text{sign } M = \text{sign } \tilde{V} + \text{sign } \tilde{W}.$$

On the other hand, it is not difficult to verify

(3) $$\tilde{L}[M] = \tilde{L}[\tilde{V}] + \tilde{L}[\tilde{W}]$$

Combining (1)~(3), we obtain

$$\varphi(P) = \text{sign } \tilde{W} - \tilde{L}[\tilde{W}].$$

But the right hand side does not depend on the resolution $\tilde{V}$. Therefore $\varphi(P)$ is well-defined. Q.E.D.

Now let $V^{(2n)}$ be a compact complex analytic variety of complex dimension $2n$. Then by Borel-Haefliger [1], it has a fundamental class

$$[V] \in H_{4n}(V; \mathbb{Z}).$$

Therefore we can define the signature of V exactly the same way as for the smooth oriented manifold. Now assume V has only rationally parallelizable isolated singular points. Then we have

Proposition 3-4.

$$\operatorname{sign} V = \widetilde{L}[V] + \sum_{P \in \Sigma V} \varphi(P),$$

where $\widetilde{L}[V]$ is the $\widetilde{L}$-genus of (V - cone neighborhood of ΣV).

Proof. Easy from the definition of $\varphi(P)$ and left to the reader.

§4. φ of the quotient singularities.

Let G be a finite subgroup of $U(2n)$ such that, G acts on $S^{4n-1} = \{z \in \mathbb{C}^{2n} \mid |z| = 1\}$ freely. Then $\mathbb{C}^{2n}/G$ is a complex space with one isolated singular point at the origin. Clearly this is a rationally parallelizable singularity. Therefore we have the Hirzebruch invariant φ for this singularity, which we write $\varphi(G)$. There we have

Theorem 4-1.

$$\varphi(G) = \frac{\operatorname{def} G + \beta_n}{|G|}$$

where $|G|$ is the order of G and

$$\operatorname{def} G = \sum_{g \in G - \{e\}} (-1)^n \cot \frac{\pi\theta_g(1)}{2} \cdots \cot \frac{\pi\theta_g(2n)}{2}$$

where

$$\exp i\pi\theta_g(1), \cdots, \exp i\pi\theta_g(2n)$$

are the eigen values of $g \in U(2n)$.

Remark. This theorem has been first obtained by Hirzebruch for $n = 1$ [8].

Proof. Let $S^{4n-1} = \{z \in \mathbb{C}^{2n} \mid |z| = 1\}$. Then S^{4n-1} has a U-structure induced from the complex structure on $\mathbb{C}^{2n}$. Clearly the action of G preserves this structure. Thus (S^{4n-1}, G) is an element of $\Omega^U_{4n-1}(G)$, the bordism group of free G U-manifolds. By a similar argument to Conner-Floyd [4], we can show that $\Omega^U_{4n-1}(G)$ is a torsion group. Thus there is a positive integer m and a free G U-manifold W such that

$$\partial(W, G) = m(S^{4n-1}, G).$$

Now consider the manifold W/G. Since the action of G preserves the U-structure of W, W/G is also a U-manifold. Let π be the subgroup $\mathrm{Im}(\pi_1(W) \longrightarrow \pi_1(W/G))$ of $\pi_1(W/G)$. Now a neighborhood of $\partial(W/G) = m(S^{4n-1}/G)$ has an almost complex structure induced from the complex structure on $\mathbb{C}^{2n} - \{0\}/G$. By Proposition 2-6, we can modify W/G to obtain an almost complex manifold $\widetilde{W/G}$ such that

(i) the almost complex structure on a neighborhood of $\partial(\widetilde{W/G}) = \partial(W/G)$ coincides with the given one.

(ii) there is a map

$$p = \widetilde{W/G} \longrightarrow W/G$$

such that

$$p_* : \pi_1(\widetilde{W/G}) \xrightarrow{\simeq} \pi_1(W/G).$$

Let $\widetilde{W}$ be the covering space of $\widetilde{W/G}$ corresponding to the subgroup $p_*^{-1}(\pi) \subset \pi_1(\widetilde{W/G})$. Then there is a free G-action on $\widetilde{W}$ such that

$$\widetilde{W}/G = \widetilde{W/G}\ .$$

Now let

$$M = \widetilde{W} \bigcup_{\partial\widetilde{W}} mD^{4n}\ .$$

Then M is an almost complex manifold, on which G acts and

$M/G = \widetilde{W}/G \cup$ cones over each connected component of the boundary.

Now since M is an almost complex manifold, we have

(1) $$\operatorname{sign} M = \widetilde{L}[M].$$

By the Atiyah-Singer G-signature theorem and the argument of Hirzebruch in [7],

(2) $$|G| \operatorname{sign} M/G = \operatorname{sign} M + m \operatorname{def} G.$$

On the other hand, by the definition of $\varphi(G)$

(3) $$\operatorname{sign} M/G = \widetilde{L}[\widetilde{W}/G] + m\,\varphi(G).$$

(cf. Proposition 3-4). Since $\pi : \widetilde{W} \longrightarrow \widetilde{W}/G$ is a $|G|$-fold cover of almost complex manifolds,

(4) $$|G|\,\widetilde{L}[\widetilde{W}/G] = \widetilde{L}[\widetilde{W}].$$

Finally we have

(5) $$\widetilde{L}[M] = \widetilde{L}[\widetilde{W}] + m\,\beta_n .$$

Combining (1)$\sim$(5), we obtain the theorem. Q.E.D.

Example 4-2. Let $\zeta = \exp \frac{2\pi i}{p}$ and let $q_1, q_2, \cdots, q_{2n}$ be natural numbers such that $(q_j, p) = 1$ for all j. Let

$$G = \{ (\zeta^{q_1}, \zeta^{q_2}, \cdots, \zeta^{q_{2n}})^i \mid i \in Z \} \subset T^{2n} \subset U(2n).$$

Then clearly G acts on S^{4n-1} freely and S^{4n-1}/G is the lens space of type $(p; q_1, q_2, \cdots, q_{2n})$. Then by Theorem 4-1, we have

$$\varphi(G) = \frac{\operatorname{def}(p;\, q_1, q_2, \cdots, q_{2n}) + \beta_n}{p}$$

where $\operatorname{def}(p; q_1, q_2, \cdots, q_{2n})$ is the number defined in [7].

§ 5. Some special cases.

Let $(V^{(2n)}, P)$ be a germ of complex analytic space at P and assume that the "boundary" of V, ∂V, is diffeomorphic to a homotopy sphere Σ^{4n-1}, which bounds a parallelizable manifold. Then the values of our invariant $\varphi(P)$ are restricted as the following proposition states.

Proposition 5-1. Let $(V^{(2n)}, P)$ be a germ of complex analytic variety at P and assume

(i) P is an isolated singular point.

(ii) ∂V is diffeomorphic to a homotopy sphere Σ^{4n-1}, which bounds a parallelizable manifold with signature m.

(iii) $n \equiv 1 \pmod 2$.

Then

$$\varphi(P) = 2\beta_n \cdot a - \beta_n + m$$

for some $a \in \mathbb{Z}$.

Proof. First assume that Σ^{4n-1} is a natural sphere S^{4n-1}. Let $\pi : \widetilde{V} \longrightarrow V$ be a resolution. Then since $\partial \widetilde{V} = S^{4n-1}$, $M = \widetilde{V} \cup$ cone over $\partial \widetilde{V}$ admits a smooth structure. Moreover since $\pi_{4n-1}(SO/U) = 0$, M admits a U-structure which is stably equivalent to the complex structure on $\widetilde{V}$. Now by definition,

$$\varphi(P) = \widetilde{L}[\widetilde{V}] - \operatorname{sign} \widetilde{V}.$$

Clearly

$$\operatorname{sign} \widetilde{V} = \operatorname{sign} M.$$

Since M is a U-manifold, we have

$$\operatorname{sign} M = L[M].$$

Hence

$$\varphi(P) = \widetilde{L}[\widetilde{V}] - L[M] .$$

Now since all the decomposable Chern numbers of $\widetilde{V}$ and M coincide, we have

$$\varphi(P) = \beta_n \chi(\widetilde{V}) - \beta_n c_n[M]$$

$$= \beta_n(\chi(M) - 1) - \beta_n c_n[M]$$
$$= 2\beta_n \lambda(M) - \beta_n.$$

Since $\lambda(M)$ is an integer, we have proved the proposition for the case $\Sigma^{4n-1} = S^{4n-1}$.

Next we consider the general case. Thus let $\pi : \widetilde{V} \longrightarrow V$ be a resolution with $\partial V = \Sigma^{4n-1}$, which bounds a parallelizable manifold W with sign $W = m$. Consider the manifold $-W$ (orientation reversed). Since $-W$ is parallelizable, it has a trivial almost complex structure. Now consider the manifold $\widetilde{V} \natural (-W)$ ($\natural$ denotes the boundary connected sum). It admits an almost complex structure compatible with the original structures on $\widetilde{V}$ and $-W$. Now $\partial(\widetilde{V} \natural (-W)) = \Sigma \# - \Sigma = S^{4n-1}$. Let

$$M = \widetilde{V} \natural (-W) \cup \text{cone over the boundary.}$$

Then as before M admits a U-structure which is stably equivalent to the almost complex structure on $\widetilde{V} \natural (-W)$. Now

$$\varphi(P) = \widetilde{L}[\widetilde{V}] - \text{sign } \widetilde{V}.$$

On the other hand, we have

$$\widetilde{L}[\widetilde{V} \natural (-W)] = \widetilde{L}[\widetilde{V}] + \beta_n \chi(W) - \beta_n$$

and

$$\text{sign } M = \text{sign } \widetilde{V} + \text{sign } (-W)$$
$$= \text{sign } \widetilde{V} - m.$$

Therefore we have

$$\varphi(P) = \widetilde{L}[\widetilde{V} \natural (-W)] + \beta_n - \text{sign } M - \beta_n \chi(W) + m$$
$$= \widetilde{L}[M] - \text{sign } M - \beta_n \chi(W) + m$$
$$= 2\beta_n \lambda(M) - \beta_n \chi(W) + m .$$

Now consider $\hat{W} = W \cup$ cone over ∂W. Then sign $\hat{W} = m$, which is divisible by 8. Since $\operatorname{sign} \hat{W} \equiv \chi(\hat{W}) \pmod 2$, we have $\chi(\hat{W}) \equiv 0 \pmod 2$. But $\chi(\hat{W}) = \chi(W) + 1$. Hence $\chi(W) \equiv 1 \pmod 2$. Therefore

$$\varphi(P) = 2\beta_n a - \beta_n + m$$

for some $a \in \mathbb{Z}$, which completes the proof. Q. E. D.

Remark 5-2. In case $n \equiv 0 \pmod 2$, I do not know whether Proposition 5-1 holds or not. In this case $\pi_{4n-1}(SO/U) = \mathbb{Z}/2$. It is easy to define an invariant $\alpha(P) \in \mathbb{Z}/2$ to detect this group. However, in general, it is difficult to calculate $\alpha(P)$. But if (V, P) is a hypersurface isolated singularity, then $\alpha(P) = 0$ and Proposition 5-1 holds.

Remark 5-3. Let $(V^{(2n)}, P)$ be the Brieskorn singularity defined by

$$z_1^{a_1} + z_2^{a_2} + \cdots + z_{2n}^{a_{2n}} = 0.$$

Then

$$\varphi(P) = \beta_n(\mu + 1) - \operatorname{sign} V_{\mathbf{a}}$$

where μ is the middle Betti number of the non-singular hypersurface $V_{\mathbf{a}} = \left\{ z_1^{a_1} + \cdots + z_{2n}^{a_{2n}} = 1 \right\}$.

Bibliography

[1] Borel, A., Haefliger, A., La classe d'homologie fundamentale d'un espace analytique, Bull. Soc. Math. France 89 (1961), 461-513.

[2] Borel, A. and Serre, J. P., Groupes de Lie et puissances réduites de Steenrod, Amer. J. Math. 75 (1953), 409-448.

[3] Browder, W., The Kervaire invariant of framed manifolds and its generalization, Ann. of Math. 90 (1969), 157-186.

[4] Conner, P. E. and Floyd, E. E., Differentiable periodic maps, Springer Verlag, 1964.

[5] Hironaka, H., Desingularization of complex-analytic varieties, Actes, Congrès intern. Math., 1970, Tome 2, 627-631.

[6] Hirzebruch, F., Topological methods in algebraic geometry, third enlarged edition, 1966.

[7] ——— , The signature theorem : Reminiscences and recreation in "Prospects in mathematics", Ann. of Math. Study 70, 1971, 3-31.

[8] ——— , The Hilbert modular group, Tokyo I.M.U. Lecture.

[9] Lojasiewicz, Triangulation of semi-analytic sets, Ann. Scuola Norm. Sup. Pisa, Sc. Fis. Mat. Ser. 3, 18 (1964), 449-474.

[10] Massey, W. S., Obstructions to the existence of almost complex structures, Bull. Amer. Math. Soc. 67 (1961), 559-564.

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